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DECOMPOSITIONS OF FINITE-TO-ONE FACTOR MAPS

BY

PAUL TROW

Department of Mathematical Sciences
The University of Memphis, Memphis, Tennessee 38152, USA
e-mail: trowp@hermes.msci.memst.edu

ABSTRACT

We examine the question of when a finite-to-one factor map $\theta : \Sigma_A \to S$, from a shift of finite type onto a sofic shift, can be decomposed as a left closing map (onto a shift of finite type) followed by a right closing map (or vice versa). We give a finite procedure for deciding this question. In general, we show that there are finitely many such decompositions, up to conjugacy. If the degree of θ is one, then there is at most one such decomposition.

1. Introduction

The structure of factor maps between shifts of finite type and sofic shifts is not very well understood at present. For example, given a finite-to-one factor map θ: $\Sigma_A \to S$, from an irreducible shift of finite type onto a sofic shift, it is not known how to decide whether θ can be written as a composition of factor maps, $\Sigma_A \to \Sigma_{B_1} \to \ldots \to \Sigma_{B_n} \to S$, each of which is right or left closing, and such that the intermediate shifts are of finite type (see [BMT] for definitions). Not all factor maps can be decomposed in this way: B. Kitchens has given examples of maps which cannot be decomposed into closing maps (see [K] or [KMT, Section 3]). But for an arbitrary map θ, the decision problem is open. In general, it is not known whether the number of conjugacy classes of decompositions of a finite-to-one factor map (without any closing assumptions on the intermediate maps) must always be finite.

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In this paper, we examine a special case of the first problem: whether a given finite-to-one factor map $\theta\colon \Sigma_A \to S$ decomposes into a left closing map (onto a shift of finite type) followed by a right closing map (or vice versa). We give a finite procedure for finding all such decompositions (Theorem 5.4). In general, we show that there are only finitely many conjugacy classes of such decompositions (Corollary 4.9). If the degree of θ is one, then any two such decompositions are conjugate, and the second map is the induced right closing cover (or left closing cover), defined by Nasu [N]. If the degree of θ is greater than one, then the second map is a conjugate to one of a finite collection of maps, which are generalizations of the induced right (or left) closing covers, and which are determined by the map θ (Theorem 4.7).

In proving Theorem 5.4, we give a finite procedure for the following problem: given a factor map $\theta\colon \Sigma_A \to S$ (not necessarily finite-to-one) and a finite-to-one factor map $\pi\colon \Sigma_B \to S$, find all continuous, shift-commuting maps $\rho\colon \Sigma_A \to \Sigma_B$ such that $\pi\rho = \theta$ (see Corollary 5.3).

In Section 3, we introduce generalizations of the induced right closing cover and left closing cover, for maps of degree greater than one (Definition 3.5). These involve the notion of a congruence partition $\mathcal{P}$, and its induced right closing cover $\pi_{r(\theta,\mathcal{P})}\colon \Sigma_{R(\theta,\mathcal{P})} \to S$. For maps of degree one, this cover is just Nasu's induced right closing cover.

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2. Background

We assume that the reader has some familiarity with symbolic dynamics. For further backround, see [BMT], [AM] or [W].

A shift of finite type is defined by a non-negative, integral matrix A. If A is $n \times n$, A defines a directed graph $G(A)$ on n vertices (or states), with $A_{i,j}$ edges from state i to state j. We denote the set of edges of $G(A)$ by $\mathcal{E}_A$ and the set of states of $G(A)$ by $\mathcal{S}_A$. Elements of $\mathcal{E}_A$ are called **symbols**. If $e \in \mathcal{E}_A$, we denote the initial state of e by $\iota(e)$ and the terminal state of e by $\tau(e)$. The **shift of finite type** Σ_A is defined by $\Sigma_A = \{x \in \mathcal{E}_A^{\mathbb{Z}} | \ \tau(x_i) = \iota(x_{i+1}) \text{ for all all } i \in \mathbb{Z}\}$. An A-**word** is any finite word $x = x_1 \dots x_k$ which appears in some $x \in \Sigma_A$. We say that x **begins at** i if $\iota(x_1) = i$, and write $\iota(x) = i$; similarly, we say that

ends at j if $\tau(x_k) = j$, and write $\tau(x) = j$.

We define an equivalence relation on $\mathcal{S}_A$ by $i \sim j$ if there exist directed paths in $G(A)$ from i to j and from j to i. For each equivalence class E, the subgraph of $G(A)$ induced by restricting to E is called an **irreducible component** of $G(A)$ [AM, Prop. 3.22]. Let C denote the transition matrix of a component. The shift of finite type Σ_C is called an **irreducible component** of Σ_A. Since $G(A)$ is finite, there are only finitely many irreducible components. If there is just one component, we say that Σ_A is **irreducible**. For the remainder of this paper, except in Section 5, we will assume that all shifts of finite type are irreducible.

If X is any symbolic dynamical system, the **shift** is the homeomorphism σ: $X \to X$, defined by $(\sigma(x))_i = x_{i+1}$. A **factor map** is a continuous, shift-commuting map from one symbolic dynamical system onto another. If the map is a homeomorphism, it is a **topological conjugacy**. A symbolic dynamical system S is called a **sofic shift** if it is the image of a factor map θ: $\Sigma_A \to S$, for some shift of finite type Σ_A. The symbols of S are denoted $\mathcal{E}_S$. An S-**word** is any finite word $w = w_1 \cdots w_k$ which appears in some point in S. We say that w is **allowed**. The **follower set** of w, denoted $\mathcal{F}(w)$, is the set of S-words u such that wu is allowed.

A factor map θ: $\Sigma_A \to S$ is a $2N+1$-**block map** if there exists a map $\tilde{\theta}$: $\mathcal{E}_A^{2N+1} \to \mathcal{E}_S$, for some non-negative integer N, such that for all $x \in \Sigma_A$, we have $\theta(x)_i = \tilde{\theta}(x_{i-N} \cdots x_{i+N})$ for all $i \in \mathbb{Z}$. By [He, Theorem 3.4], any factor map has such a representation, for some N. We let θ denote both the factor map and the map $\tilde{\theta}$ on finite words. Up to topological conjugacy of the domain, any factor map θ can be assumed to be a one-block map [KMT, p. 86]. A one-block map θ defines a labelling of the graph $G(A)$, where $e \in \mathcal{E}_A$ is labelled $\theta(e)$. If $x = x_1 \cdots x_k$ is an A -word, then $\theta(x)$ is the S-word $\theta(x_1) \ldots \theta(x_k)$. If w and v are S-words and wv is allowed, we write $ab \in \theta^{-1}(wv)$ if a and b are A-words such that ab is allowed, $\theta(a) = w$ and $\theta(b) = v$.

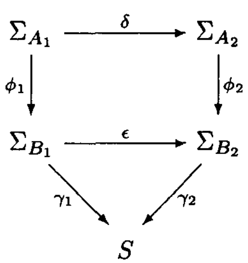
If θ: $\Sigma_A \to \Sigma_B$ is a factor map between shifts of finite type, we define the **induced map on states** $\hat{\theta}$: $\mathcal{S}_A \to \mathcal{S}_B$ by $\hat{\theta}(i) = j$ if there exist $e \in \mathcal{E}_A$, $f \in \mathcal{E}_B$, with $\iota(e) = i$ and $\iota(f) = j$, such that $\theta(e) = f$.

A factor map θ: $\Sigma_A \to S$ is **finite-to-one** if there exists $N \in \mathbb{Z}^+$ such that $\#\theta^{-1}(y) \leq N$ for all $y \in S$ (where $\#$ denotes the cardinality of a set). It is known that θ is finite-to-one if and only if $h(\Sigma_A) = h(S)$, where h denotes the topological entropy [CP2, Corollary 4.4].

A one-block factor map $\theta\colon \Sigma_A \to S$ is **right resolving** if the following condition holds: if $e_1 \in \mathcal{E}_A$, $f_1 \in \mathcal{E}_S$, $\theta(e_1) = f_1$ and $f_1 f_2$ is an allowed S-word, then there exists at most one $e_2 \in \mathcal{E}_A$ such that $e_1 e_2$ is allowed and $\theta(e_2) = f_2$. If S is a shift of finite type, then the words *at most one* can be replaced by a *unique* [BMT, p.8]. The definition of **left resolving** is similar, the condition being $\theta(e_2) = f_2$ and $f_1 f_2$ allowed implies there exists at most one e_1 such that $e_1 e_2$ is allowed and $\theta(e_1) = f_1$.

A factor map $\theta\colon \Sigma_A \to S$ is **right closing** if θ does not identify any two distinct points x and y with the property that there exists $N \in \mathbb{Z}$ such that $x_i = y_i$ for all $i \leq N$. The definition of **left closing** is similar: simply replace $i \leq N$ with $i \geq N$. Any right or left closing map is finite-to-one [BMT, Prop. 2.2].

Two factor maps $\theta_1\colon \Sigma_{A_1} \to S$ and $\theta_2\colon \Sigma_{A_2} \to S$ are **conjugate** if there exists a topological conjugacy $\beta\colon \Sigma_{A_1} \to \Sigma_{A_2}$ such that $\theta_2 = \theta_1\beta$. A **decomposition** of $\theta\colon \Sigma_A \to S$ is a pair of factor maps $\phi\colon \Sigma_A \to \Sigma_B$ and $\gamma\colon \Sigma_B \to S$, where Σ_B is some shift of finite type, such that $\theta = \gamma\phi$. Two decompositions $\phi_1\colon \Sigma_{A_1} \to \Sigma_{B_1}$, $\gamma_1\colon \Sigma_{B_1} \to S$ and $\phi_2\colon \Sigma_{A_2} \to \Sigma_{B_2}$, $\gamma_2\colon \Sigma_{B_2} \to S$ of $\theta_1 = \gamma_1\phi_1$ and $\theta_2 = \gamma_2\phi_2$, respectively, are **conjugate** if there exist topological conjugacies $\delta\colon \Sigma_{A_1} \to \Sigma_{A_2}$ and $\epsilon\colon \Sigma_{B_1} \to \Sigma_{B_2}$ which makes the following diagram commute.



We next recall the definitions of the degree of a factor map and magic words.

Definition 2.1: Let Σ_A be an irreducible shift of finite type, S and a sofic shift and $\theta\colon \Sigma_A \to S$ a finite- to-one one-block factor map. The **degree** of θ, denoted $deg(\theta)$, is defined by

$$\deg(\theta) = \inf_{\substack{y_1\ldots y_n \\ \text{an S-word}}} \ \inf_{1 \leq i \leq n} \# \left\{ a \;\middle|\; \begin{array}{l} \text{there exists an } A\text{-word} \\ x_1 \cdots x_n \in \theta^{-1}(y_1 \ldots y_n) \\ \text{with } x_i = a \end{array} \right\}.$$

An S-word $m_1 \ldots m_n$ for which the infimum occurs is called a **magic word** for θ, and the symbol m_s, $1 \leq s \leq n$, at which the infimum occurs is called a **magic coordinate**.

The degree of any finite-to-one factor map $\theta\colon \Sigma_A \to S$ can be defined to be the degree of any recoding of θ to a one-block map (which is independent of the recoding).

Definition 2.2: For a given finite-to-one one-block factor map θ and magic word m, with magic coordinate m_s, we define $S(\theta, m) = \{\alpha \in \mathcal{E}_A |$ there exists $a \in \theta^{-1}(m)$ with $a_s = \alpha\}$. By definition, $deg(\theta) = \#S(\theta, m)$.

If umv is an S-word, then it is clear that umv is a magic word for θ, with magic coordinate m_s, and that $S(\theta, umv) = S(\theta, m)$.

The following Lemma is essentially contained in [CP1].

LEMMA 2.3: *Let Σ_A be an irreducible shift of finite type, S a sofic shift and $\theta\colon \Sigma_A \to S$ a finite-to-one, one-block factor map. Let m be a magic word for θ, with magic coordinate m_s. Let w be an S-word such that mwm is allowed. Then for each $\alpha \in S(\theta, m)$, there exists an A-word $abc \in \theta^{-1}(mwm)$ such that $a_s = \alpha$. If abc, $a'b'c' \in \theta^{-1}(mwm)$, then $a_s = a'_s$ if and only if $c_s = c'_s$.*

Proof: The proof of Lemma 2.3 is quite similar to the proof of [KMT, Lemma 2.4 (2), p. 93], to which we refer the reader for details. The proof in [KMT] assumes that m is a magic symbol, i.e. a magic word of length 1; however, it is not difficult to modify the details of that proof to establish Lemma 2.3. ∎

By Lemma 2.3, for each $\alpha \in S(\theta, m)$, there exists a unique $\alpha' \in S(\theta, m)$ such that if $abc \in \theta^{-1}(mwm)$ with $a_s = \alpha$, then $c_s = \alpha'$.

Definition 2.4: Let θ and m be as in Lemma 2.3. We define a mapping $\Gamma^{\theta}_{mwm}\colon S(\theta, m) \to S(\theta, m)$ by $\Gamma^{\theta}_{mwm}(\alpha) = \alpha'$, where α' is the unique element of $S(\theta, m)$ corresponding to $\alpha \in S(\theta, m)$, whose existence is guaranteed by Lemma 2.3.

If there is only one map θ being considered, we omit the superscript θ and write $\Gamma^{\theta}_{mwm} = \Gamma_{mwm}$. By Lemma 2.3, Γ_{mwm} is injective, and therefore it is a bijection.

The following lemma is easy to show, using Lemma 2.3. The proof is omitted.

LEMMA 2.5: *If w and v are S-words such that mwm and mvm are allowed, then $\Gamma_{mwmvm} = \Gamma_{mvm}\Gamma_{mwm}$.*

LEMMA 2.6: *If w is an S-word such that mwm is allowed, then there is an S-word $\bar{w}$ such that $m\bar{w}m$ is allowed, and $\Gamma_{m\bar{w}m} = \Gamma_{mwm}^{-1}$.*

Proof: Since, by Lemma 2.5, $\{\Gamma_{mwm} |\ mwm \text{ is allowed}\}$ is a subset of the symmetric group on $S(\theta, m)$ which is closed under products, the result follows by elementary group theory. ∎

It follows from Lemmas 2.5 and 2.6 that $\{\Gamma_{mwm} |\ mwm \text{ is allowed}\}$ forms a subgroup of the symmetric group on $S(\theta, m)$. This group has been studied in a number of other contexts; for example, see [S].

The following two lemmas are well-known in symbolic dynamics.

LEMMA 2.7: *Let Σ_A and Σ_B be irreducible shifts of finite type, S a sofic shift and let ϕ: $\Sigma_A \to \Sigma_B$ and γ: $\Sigma_B \to S$ be finite-to-one factor maps. Let $\theta = \gamma\phi$. Then $\deg(\theta) = \deg(\gamma)\deg(\phi)$.*

Proof: Let $z \in S$ be a bilaterally transitive point. Then by [CP1, Theorems 6.4 and 6.9], z has exactly $\deg(\gamma)$ preimages under γ, each of which is bilaterally transitive. Again, by [CP1, Theorem 6.4], each of these preimages has exactly $\deg(\phi)$ preimages under ϕ. Since z has exactly $\deg(\theta)$ preimages under θ, it follows that $\deg(\theta) = \deg(\gamma)\deg(\phi)$. ∎

In Lemmas 2.8, 2.9 and 2.11, and Definition 2.10, we assume that Σ_A and Σ_B are irreducible shifts of finite type, S is a sofic shift and that ϕ: $\Sigma_A \to \Sigma_B$ and γ: $\Sigma_B \to S$ are finite-to-one, one-block factor maps. Let $\theta = \gamma\phi$. Let m be a magic word for θ, with magic coordinate m_s.

LEMMA 2.8: *The word m is magic for γ, with magic coordinate m_s. If $n \in \gamma^{-1}(m)$, then n is a magic word for ϕ, with magic coordinate n_s.*

Proof: Let $Y = \{\alpha |\ \text{there exists } b \in \gamma^{-1}(m) \text{ with } b_s = \alpha\}$. Then $\#Y \geq \deg(\gamma)$, by the minimality condition in the definition of degree. For each $\alpha \in Y$, choose $b^\alpha \in \gamma^{-1}(m)$ such that $b^\alpha_s = \alpha$. We may assume that n is one of the b^α. For each b^α, let $Z^\alpha = \{\alpha' |\ \text{there exists } a \in \phi^{-1}(b^\alpha) \text{ with } a_s = \alpha'\}$. Clearly, the sets Z^α are disjoint, $Z^\alpha \subseteq S(\theta, m)$ and $\#Z^\alpha \geq \deg(\phi)$ for each α. Since $\bigcup_{\alpha\in Y} Z^\alpha \subseteq S(\theta, m)$, we have $\#\bigcup_{\alpha\in Y} Z^\alpha \leq \#S(\theta, m) = \deg(\theta)$. On the other hand, $\#\bigcup_{\alpha\in Y} Z^\alpha \geq \#Y(\deg(\phi)) \geq \deg(\gamma)\deg(\phi) = \deg(\theta)$ (by Lemma 2.7). It follows that $\#Y = \deg(\gamma)$, and so m is magic for γ, and has magic coordinate

m_s. Similarly, $\#Z^\alpha = \deg(\phi)$ for each α, so n is magic for ϕ, with magic coordinate n_s. ∎

It follows from Lemma 2.8 that $S(\gamma, m)$ can be defined as in Definition 2.4.

LEMMA 2.9: *The map Λ_ϕ: $S(\theta, m) \to S(\gamma, m)$ defined by $\Lambda_\phi(\alpha) = \phi(\alpha)$ is surjective. If $deg(\phi) = 1$, then Λ_ϕ is a bijection.*

Proof: If $\alpha \in S(\theta, m)$, then there exists $\bar{a} \in \theta^{-1}(m)$ with $\bar{a}_s = \alpha$. Then $a = \phi(\bar{a}) \in \gamma^{-1}(m)$ and $\phi(\alpha) = \phi(\bar{a}_s) = a_s \in S(\gamma, m)$. So Λ_ϕ maps $S(\theta, m)$ into $S(\gamma, m)$. To see that Λ_ϕ is surjective, let $\alpha \in S(\gamma, m)$. Then there exists $a \in \gamma^{-1}(m)$ such that $a_s = \alpha$. Let $\bar{a} \in \phi^{-1}(a)$, so that $\bar{a} \in \theta^{-1}(m)$. Since $\bar{a}_s \in S(\theta, m)$ and $\phi(\bar{a}_s) = a_s = \alpha$, it follows that Λ_ϕ is surjective. Finally, suppose that $deg(\phi) = 1$. Since $\#S(\gamma, m) = deg(\gamma)$ and $\#S(\theta, m) = deg(\theta) = deg(\gamma)deg(\phi) = deg(\gamma)$, by Lemma 2.7, it follows that Λ_ϕ is injective. ∎

Definition 2.10: If $\mathcal{P}$ is a partition of $S(\gamma, m)$, then $\Lambda_\phi^{-1}(\mathcal{P}) = \{\Lambda_\phi^{-1}(P) \mid P \in \mathcal{P}\}$ is a partition of $S(\theta, m)$. If $\mathcal{P}$ is the partition of $S(\gamma, m)$ into singleton sets, then $\Lambda_\phi^{-1}(\mathcal{P})$ is called the **partition induced by** $\gamma\phi$.

Clearly, α and α' are in the same partition element of $\Lambda_\phi^{-1}(\mathcal{P})$ if and only if $\phi(\alpha) = \phi(\alpha')$. It is not hard to see that for each $P \in \Lambda_\phi^{-1}(\mathcal{P})$, we have $\#P = deg(\phi)$ and that the number of partition elements equals $\deg(\gamma)$.

LEMMA 2.11: *Let w be an S-word such that mwm is allowed. Then the following diagram commutes.*

$$\begin{array}{ccc} S(\theta, m) & \xrightarrow{\Gamma^\theta_{mwm}} & S(\theta, m) \\ {\scriptstyle \Lambda_\phi}\downarrow & & \downarrow{\scriptstyle \Lambda_\phi} \\ S(\gamma, m) & \xrightarrow{\Gamma^\gamma_{mwm}} & S(\gamma, m) \end{array}$$

Proof: Let $\alpha \in S(\theta, m)$. By Lemma 2.3, there is a word $\bar{a}\bar{b}\bar{c} \in \theta^{-1}(mwm)$ such that $\bar{a}_s = \alpha$. Then $\Gamma^\theta_{mwm}(\alpha) = c_s$. Let $\phi(\bar{a}\bar{b}\bar{c}) = \phi(abc)$, so that $abc \in \gamma^{-1}(mwm)$, since $\theta = \gamma\phi$. Then $a_s \in S(\gamma, m)$ and $\Gamma^\gamma_{mwm}(a_s) = c_s$. Therefore $\Lambda_\phi(\Gamma^\theta_{mwm}(\alpha)) = \phi(\bar{c}_s) = c_s$, while $\Gamma^\gamma_{mwm}(\Lambda_\phi(\alpha)) = \Gamma^\gamma_{mwm}(\phi(\bar{a}_s)) = \Gamma^\gamma_{mwm}(a_s) = c_s$, and so the above diagram commutes. ∎

If θ: $\Sigma_A \to S$ and θ_1: $\Sigma_{A_1} \to S$ are conjugate one-block factor maps, we would like to define an equivalence between partitions of $S(\theta, m)$ and partitions of $S(\theta_1, m)$. One problem with this is that words which are magic for θ are not necessarily magic for θ_1. To resolve this problem, choose a magic word m for θ, with magic coordinate m_s. Now choose a left infinite ray ${}^{\leftarrow}u = \cdots u_3u_2u_1$ and a right infinite ray $v^{\rightarrow} = v_1v_2v_3\cdots$, both of which contain infinitely many occurences of m, and such that ${}^{\leftarrow}umv^{\rightarrow}$ is an allowed S-word. If $u = u_j\cdots u_1$ and $v = v_1\cdots v_k$ are initial segments of ${}^{\leftarrow}u$ and $v^{\rightarrow}$, respectively, we define an **extension** of umv to be a word of the form $u_{j'}\cdots u_1mv_1\cdots v_{k'}$, where $j' \geq j$ and $k' \geq k$. Now, by going to higher block presentations, there is a shift of finite type $\Sigma_{\bar{A}}$ and one-block conjugacies β: $\Sigma_{\bar{A}} \to \Sigma_A$ and β_1: $\Sigma_{\bar{A}} \to \Sigma_{A_1}$ such that $\theta\beta = \theta_1\beta_1 = \bar{\theta}$. If we choose an extension umv of m, such that u begins with m and v ends with m, then any preimage of uwv under θ will contain one of $deg(\theta)$ mutually separated words between the coordinates corresponding to m_s in the first and last occurences of m in umv (by Lemma 2.3). Since β is a conjugacy, it follows that if u and v are sufficiently long then $\bar{m} = umv$ is a magic word for $\bar{\theta}$, with magic coordinate m_s. By Lemma 2.8, $\bar{m}$ is magic for θ and θ_1, with magic coordinate m_s. Clearly, $S(\theta,\bar{m}) = S(\theta,m)$, and we may define Λ_β: $S(\bar{\theta},\bar{m}) \to S(\theta,\bar{m}) = S(\theta,m)$ and Λ_{β_1}: $S(\bar{\theta},\bar{m}) \to S(\theta_1,\bar{m})$, which are bijections, by Lemma 2.9. Also, if θ_2: $\Sigma_{A_2} \to S$ is another one-block factor map which is conjugate to θ, then we can choose a still longer extension $\tilde{m}$ which is a magic word for θ_1 and θ_2. We can now make the following definition.

Definition 2.12: Let θ_1: $\Sigma_{A_1} \to S$ and θ_2: $\Sigma_{A_2} \to S$ be finite-to-one, one-block factor maps, both of which are conjugate to θ, and let $\tilde{m}$ be an extension of m which is magic for θ_1 and θ_2. If $\mathcal{P}_1$ and $\mathcal{P}_2$ are partitions of $S(\theta_1,\tilde{m})$ and $S(\theta_2,\tilde{m})$, respectively, we say that $\mathcal{P}_1$ is **equivalent** to $\mathcal{P}_2$ if there is a shift of finite type $\Sigma_{\bar{A}}$ and one-block conjugacies β_1: $\Sigma_{\bar{A}} \to \Sigma_{A_1}$ and β_2: $\Sigma_{\bar{A}} \to \Sigma_{A_2}$, such that $\theta_1\beta_1 = \theta_2\beta_2 = \bar{\theta}$, and an extension $\bar{m}$ of $\tilde{m}$ which is magic for $\bar{\theta}$, such that $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1) = \Lambda_{\beta_2}^{-1}(\mathcal{P}_2)$.

This definition is easily seen to give an equivalence relation on partitions of sets of the form $S(\theta_1,\bar{m})$, where θ_1 is a one-block factor map which is conjugate to θ and $\bar{m}$ is an extension of m which is magic for θ_1. We denote the equivalence class of a partition by $[\mathcal{P}]$. When the meaning is clear, we will use the term *partition* to refer to an equivalence class of partitions.

LEMMA 2.13: *Let θ_1: $\Sigma_{A_1} \to S$ be a one-block factor map which is conjugate to θ, $\bar{m}$ an extension of m which is magic for θ and θ_1, and $\mathcal{P}_1$ a partition of $S(\theta_1,\bar{m})$. There exists a partition $\mathcal{P}$ of $S(\theta,\bar{m})$ which is equivalent to $\mathcal{P}_1$.*

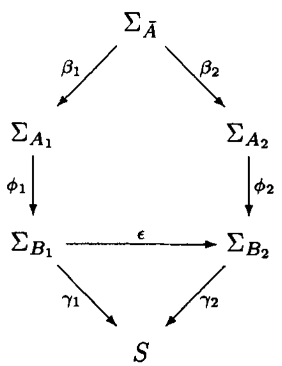
Proof: Just set $\mathcal{P} = \Lambda_\beta \Lambda_{\beta_1}^{-1}(\mathcal{P}_1)$, where β and β_1 are as in the remarks preceding Definition 2.12. ∎

It follows from Lemma 2.13 that there are only finitely many equivalence classes of partitions of sets of the form $S(\theta_1, \bar{m})$, since any partition of $S(\theta_1, \bar{m})$ is equivalent to a partition of $S(\theta, m)$, and there are only finitely many partitions of the finite set $S(\theta, m)$.

From now on we will make the standing assumption that a magic word for a factor map $\theta_1\colon \Sigma_{A_1} \to S$, conjugate to a given factor map θ, is an extension of the fixed magic word m for θ. We next show that conjugate decompositions of θ induce equivalent partitions.

PROPOSITION 2.14: *Let $\phi_1\colon \Sigma_{A_1} \to \Sigma_{B_1}$, $\gamma_1\colon \Sigma_{B_1} \to S$, $\phi_2\colon \Sigma_{A_2} \to \Sigma_{B_2}$ and $\gamma_2\colon \Sigma_{B_2} \to S$ be finite-to-one, one-block factor maps, and suppose that $\gamma_1\phi_1$ and $\gamma_2\phi_2$ are conjugate decompositions of θ. Let $\theta_1 = \gamma_1\phi_1$ and $\theta_2 = \gamma_2\phi_2$, and let m be a magic word for θ_1 and θ_2. Let $\mathcal{P}_1$ be the partition induced by $\gamma_1\phi_1$ and $\mathcal{P}_2$ the partition induced by $\gamma_2\phi_2$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent partitions.*

Proof: Since $\gamma_1\phi_1$ and $\gamma_2\phi_2$ are conjugate decompositions, it follows by the remarks preceding Definition 2.12 that there is a topological conjuacy $\epsilon\colon \Sigma_{B_1} \to \Sigma_{B_2}$, a shift of finite type $\Sigma_{\bar{A}}$ and one-block conjugacies $\beta_1\colon \Sigma_{\bar{A}} \to \Sigma_{A_1}$ and $\beta_2\colon \Sigma_{\bar{A}} \to \Sigma_{A_2}$ which make the following diagram commute.



The proof now proceeds by diagram chasing. Let $\bar{\theta} = \gamma_1\phi_1\beta_1 = \gamma_2\phi_2\beta_2$. By the remarks preceeding Definition 2.12, there is an extension $\bar{m}$ of m which is magic for $\bar{\theta}$. To simplify notation, let us denote $\bar{m}$ by m. Let $\Lambda_{\beta_1}\colon S(\bar{\theta}, m) \to S(\theta_1, m)$ and $\Lambda_{\beta_2}\colon S(\bar{\theta}, m) \to S(\theta_2, m)$ be the bijections given by Lemma 2.9. We claim that $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1) = \Lambda_{\beta_2}^{-1}(\mathcal{P}_2)$, which will prove the proposition. To see the claim, observe that if α and α' are in the same partition element of $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1)$, then $\beta_1(\alpha)$ and $\beta_1(\alpha')$ are in the same partition element of $\mathcal{P}_1$, by definition of Λ_{β_1}, and so $\phi_1\beta_1(\alpha) = \phi_1\beta_1(\alpha')$, by definition of $\mathcal{P}_1$. We assume that ϵ is a $2N + 1$-block map, so that $(\epsilon(x))_i = \epsilon(x_{i-N}\dots x_{i+N})$, for $x \in \Sigma_{B_1}$. Choose an extension umv of m, where u and v each have length N. Then umv is magic for $\bar{\theta}$, with magic coordinate m_s. By Lemma 2.8, umv is magic for γ_1, with magic coordinate m_s, so we may choose $abc \in \gamma_1^{-1}(umv)$ such that $b_s = \phi_1\beta_1(\alpha) = \phi_1\beta_1(\alpha')$. By Lemma 2.8, abc is magic for $\phi_1\beta_1$, so we may choose def and $d'e'f' \in (\phi_1\beta_1)^{-1}(abc)$ such that $e_s = \alpha$ and $e'_s = \alpha'$. By the commutativity of the diagram above, $\phi_2\beta_2(e) = \epsilon\phi_1\beta_1(def) = \epsilon(abc) = \epsilon\phi_1\beta_1(d'e'f') = \phi_2\beta_2(e')$. It follows that $\phi_2\beta_2(\alpha) = \phi_2\beta_2(e_s) = \phi_2\beta_2(e'_s) = \phi_2\beta_2(\alpha')$. Therefore $\beta_2(\alpha)$ and $\beta_2(\alpha')$ are in the same partition element of $\mathcal{P}_2$, and so α and α' are in the same partition element of $\Lambda_{\beta_2}^{-1}(\mathcal{P}_2)$. This shows that $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1)$ is a finer partition than $\Lambda_{\beta_2}^{-1}(\mathcal{P}_2)$. A similar argument shows that $\Lambda_{\beta_2}^{-1}(\mathcal{P}_2)$ is finer than $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1)$, so $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1) = \Lambda_{\beta_2}^{-1}(\mathcal{P}_2)$. This proves the claim, from which the proposition follows. ∎

In Theorem 4.8, we will prove the converse to this result.

Now, if $\phi\colon \Sigma_A \to \Sigma_B$ and $\gamma\colon \Sigma_B \to S$ and $\theta = \gamma\phi$, where ϕ and γ are not necessarily one-block maps, we can define the *partition induced by* $\gamma\phi$ to be the equivalence class $[\mathcal{P}]$ of the partition $\mathcal{P}$ induced by the decomposition $\bar{\phi}\colon \Sigma_{\bar{A}} \to \Sigma_{\bar{B}}$, $\bar{\gamma}\colon \Sigma_{\bar{B}} \to S$, where $\bar{\gamma}$ and $\bar{\phi}$ are any recodings of γ and ϕ to one-block maps. By Proposition 2.14, this definition is independent of the recodings.

3. Congruence partitions

In this section, we define the concept of a congruence partition for a factor map θ, and the right closing cover of S induced by the partition. This is a generalization of Nasu's induced right closing cover [N, p. 572].

In what follows, we assume that Σ_A is an irreducible shift of finite type, S a sofic shift and that $\theta\colon \Sigma_A \to S$ is a finite-to-one, one-block factor map.

Definition 3.1: Let m be a magic word for θ, with magic coordinate m_s. A partition $\mathcal{P}$ of $S(\theta, m)$ is a **θ-congruence partition** if for every $P \in \mathcal{P}$ and S-word w such that mwm is allowed, $\Gamma_{mwm}(P) = P'$ for some $P' \in \mathcal{P}$.

The term "congruence" is borrowed from automata theory. When there is just one map θ being considered, we will omit the prefix θ and call $\mathcal{P}$ simply a congruence partition. It can be shown, using Lemma 2.11, that if two partitions are equivalent, then one is a congruence partition if and only if the other is.

It is easy to see that the partition of $S(\theta, m)$ into singleton sets and the trivial partition consisting of the entire set $S(\theta, m)$ are always congruence partitions. It follows from Definition 3.1 that if P, $P' \in \mathcal{P}$, then $\#P = \#P'$. The connection between congruence partitions and decompositions of factor maps is explained by the following lemma and definition.

LEMMA 3.2: *Let Σ_A and Σ_B be irreducible shifts of finite type, S a sofic shift, and suppose that $\phi\colon \Sigma_A \to \Sigma_B$ and $\gamma\colon \Sigma_B \to S$ are one-block factor maps. Let $\theta = \gamma\phi$, and suppose that m is a magic word for θ, with magic coordinate m_s. Let $\mathcal{P}$ be a partition of $S(\gamma, m)$ and let $\bar{\mathcal{P}} = \Lambda_\phi^{-1}(\mathcal{P})$, as in Definition 2.10. If $\mathcal{P}$ is a γ-congruence partition then $\bar{\mathcal{P}}$ is a θ-congruence partition. If $deg(\phi) = 1$, then the converse holds.*

Proof: Let w be an S-word such that mwm is allowed. If $P \in \mathcal{P}$ and $\bar{P} = \Lambda_\beta^{-1}(P)$, then it follows from Lemma 2.11 and the fact that Γ^γ_{mwm} and Γ^θ_{mwm} are invertible, that $\Gamma^\theta_{mwm}(\bar{P}) = \Gamma^\theta_{mwm}(\Lambda_\phi^{-1}(P)) = \Lambda_\phi^{-1}(\Gamma^\gamma_{mwm}(P))$. If $\mathcal{P}$ is a γ-congruence partition, then $\Gamma^\gamma_{mwm}(P) \in \mathcal{P}$, so $\Gamma^\theta_{mwm}(\bar{P}) \in \bar{\mathcal{P}}$. Therefore $\bar{\mathcal{P}}$ is a θ-congruence partition. To see the last statement, assume that $deg(\phi) = 1$ and that $\bar{\mathcal{P}}$ is a θ-congruence partition. Then $\Gamma^\theta_{mwm}(\bar{P}) = \Lambda_\phi^{-1}(P')$, for some $P' \in \mathcal{P}$. So $\Lambda_\phi^{-1}(\Gamma^\gamma_{mwm}(P)) = \Lambda_\phi^{-1}(P')$. Since Λ_ϕ is invertible, by Lemma 2.9, we have $\Gamma^\gamma_{mwm}(P) = P' \in \mathcal{P}$. Therefore $\mathcal{P}$ is a γ-congruence partition. ∎

Since the partition of $S(\gamma, m)$ into singleton sets is a γ-congruence partition, it follows from Lemma 3.2 that the partition induced by $\gamma\phi$ is a θ-congruence partition (see Definition 2.10).

We next define a collection of directed graphs and covers of S induced by the collection of θ-congruence partitions.

Definition 3.3: [KMT, p. 100]. Let $\theta\colon \Sigma_A \to S$ be a finite-to-one, one-block factor map from an irreducible shift of finite type onto a sofic shift, and assume that A is $n \times n$. Let m be a magic word for θ, with magic coordinate m_s. Let $P \subseteq S(\theta, m)$. For any S-word $w = mv$ beginning with m, define a vector $l_\theta^{(w,P)} \in \mathbb{Z}^n$ by

$$(l_\theta^{(w,P))})_i = \begin{cases} 1 & \text{if there exists } ab \in \theta^{-1}(mv), \\ & \text{ending at } i, \text{ such that } a_s \in P \\ 0 & \text{otherwise.} \end{cases}$$

For any S-word $w' = vm$ ending with m, define a vector $r_\theta^{(w',P)} \in \mathbb{Z}^n$ by

$$(r_\theta^{(w',P)})_i = \begin{cases} 1 & \text{if there exists } ab \in \theta^{-1}(vm), \\ & \text{beginning at } i, \text{ such that } b_s \in P \\ 0 & \text{otherwise.} \end{cases}$$

When there is just one map θ being considered, we omit the subscipt θ and write $l^{(w,P)} = l_\theta^{(w,P)}$ and $r^{(w',P)} = l_\theta^{(w',P)}$.

If $\mathcal{P}$ is a congruence partition, and $P, P' \in \mathcal{P}$, then the set of vectors of the form $l^{(w,P)}$ coincides with the set of vectors of the form $l^{(w',P')}$ (see Lemma 6.2).

Definition 3.4: Let $\mathcal{P}$ be a θ-congruence partition and $P \in \mathcal{P}$. We define a labelled directed graph with transition matrix $R(\theta, \mathcal{P})$ as follows. The states of $G(R(\theta, \mathcal{P}))$ consist of all vectors of the form $l^{(w,P)}$, where w begins with m. For each S-symbol a such that wa is allowed, there is an edge e from $l^{(w,P)}$ to $l^{(wa,P)}$ labelled a. Similarly, we define a labelled directed graph with transition matrix $L(\theta, \mathcal{P})$, whose states consist of all vectors of the form $r^{(w',P)}$, where w' ends with m. For each S-symbol a such that aw is allowed, there is an edge from $l^{(aw,P)}$ to $l^{(w,P)}$ labelled a.

In Section 6, we prove that the graphs $G(R(\theta, \mathcal{P}))$ and $L(R(\theta, \mathcal{P}))$ do not depend on the choice of $P \in \mathcal{P}$ (Proposition 6.3). Also, if we choose an extension $\bar{m}$ of m, then $G(R(\theta, \mathcal{P}))$ is the same labelled graph whether it is defined using m or $\bar{m}$ (Proposition 6.4). The same holds for $G(L(\theta, \mathcal{P}))$. In Proposition 6.5, we show that the shifts of finite type $\Sigma_{R(\theta,\mathcal{P})}$ and $\Sigma_{L(\theta,\mathcal{P})}$ are irreducible. For a given map θ, there are finitely many graphs $G(R(\theta, \mathcal{P}))$, since there are finitely many θ-congruence partitions.

Definition 3.5: The labelling of the edges of $G(R(\theta, \mathcal{P}))$ defines a one-block factor map $\pi_{r(\theta,\mathcal{P})}\colon \Sigma_{R(\theta,\mathcal{P})} \to S$. It is easy to check that $\pi_{r(\theta,\mathcal{P})}$ is right resolving. We call $\pi_{r(\theta,\mathcal{P})}\colon \Sigma_{R(\theta,\mathcal{P})} \to S$ the **right closing cover of S induced by the partition $\mathcal{P}$**. Similarly, there is a left resolving map $\pi_{l(\theta,\mathcal{P})}\colon \Sigma_{L(\theta,\mathcal{P})} \to S$, which we call the **left closing cover induced by the partition $\mathcal{P}$**.

The maps $\pi_{r(\theta,\mathcal{P})}$ and $\pi_{l(\theta,\mathcal{P})}$ do not depend on the choice of $P \in \mathcal{P}$ in the definitions of $G(R(\theta, \mathcal{P}))$ and $L(R(\theta, \mathcal{P}))$ (see the remark following Proposition 6.3). If $\mathcal{P}$ is the partition of $S(\theta, m)$ into singleton sets, then $\pi_\theta\colon \Sigma_{R(\theta,\mathcal{P})} \to S$ is Nasu's induced right closing cover, restricted to a component of maximal entropy [N].

Our next goal is to show that equivalent partitions induce conjugate right closing covers (Corollary 3.9). In what follows, we will prove several results concerning the covers $\pi_{r(\theta,\mathcal{P})}$. For each of these results, there is a corresponding result concerning the covers $\pi_{l(\theta,\mathcal{P})}$, whose statement and proof should be evident to the reader, and which we omit.

In Lemmas 3.6 – 3.8, we assume that Σ_A and $\Sigma_{\bar{A}}$ are irreducible shifts of finite type, S a sofic shift, and that β: $\Sigma_{\bar{A}} \to \Sigma_A$ and θ: $\Sigma_A \to S$ are finite-to-one, one-block factor maps. Let $\bar{\theta} = \theta\beta$. Assume that m is a magic word for $\bar{\theta}$, with magic coordinate m_s. Let $\mathcal{P}$ be a θ-congruence partition for m. Let $\bar{\mathcal{P}} = \Lambda_\beta^{-1}(\mathcal{P})$, which is a $\bar{\theta}$-congruence partition by Lemma 3.2. Let $P \in \mathcal{P}$ and $\bar{P} = \Lambda_\beta^{-1}(P)$.

LEMMA 3.6: *Let w and v be S-words that begin with m. If $l_{\bar{\theta}}^{(w,\bar{P})} = l_{\bar{\theta}}^{(v,\bar{P})}$, then $l_\theta^{(w,P)} = l_\theta^{(v,P)}$.*

Proof: Assume that $l_{\bar{\theta}}^{(w,\bar{P})} = l_{\bar{\theta}}^{(v,\bar{P})}$. Suppose that $(l_\theta^{(w,P)})_i = 1$. Then there is an A-word $a \in \theta^{-1}(w)$ ending at i, with $a \in P$. Choose $\bar{a} \in \beta^{-1}(a)$, so that $\bar{a}_s \in \bar{P}$, and suppose that $\bar{a}$ ends at j. Since $\bar{a} \in \bar{\theta}^{-1}(w)$, we have $(l_{\bar{\theta}}^{(w,\bar{P})})_j = 1$. Since $l_{\bar{\theta}}^{(w,\bar{P})} = l_{\bar{\theta}}^{(v,\bar{P})}$, we have $(l_{\bar{\theta}}^{(v,\bar{P})})_j = 1$, so there is an $\bar{A}$-word $\bar{b} \in \theta^{-1}(v)$ ending at j, with $\bar{b}_s \in \bar{\mathcal{P}}$. Let $b = \beta(\bar{b})$. Then b ends at i, $b \in \theta^{-1}(v)$ and $b_s \in P$. Therefore $(l_\theta^{(v,P)})_i = 1$. A similar argument shows that if $(l_\theta^{(v,P)})_i = 1$ then $(l_\theta^{(w,P)})_i = 1$. Therefore $l_\theta^{(w,P)} = l_\theta^{(v,P)}$. ∎

LEMMA 3.7: *There is a right resolving factor map ν_β: $\Sigma_{R(\bar{\theta},\bar{\mathcal{P}})} \to \Sigma_{R(\theta,\mathcal{P})}$ such that $\pi_{r(\bar{\theta},\bar{\mathcal{P}})} = \pi_{r(\theta,\mathcal{P})}\nu_\beta$.*

Proof: For w an S-word beginning with m, define $\hat{\nu}_\beta$: $\mathcal{S}_{R(\bar{\theta},\bar{\mathcal{P}})} \to \mathcal{S}_{R(\theta,\mathcal{P})}$ by $\hat{\nu}_\beta(l_{\bar{\theta}}^{(w,\bar{P})}) = l_\theta^{(w,P)}$. It follows from Lemma 3.6 that $\hat{\nu}_\beta$ is well-defined. We observe that for any $l_{\bar{\theta}}^{(w,\bar{P})}$, there is a one-to-one correspondence ν_β between $\{e \in \mathcal{E}_{R(\bar{\theta},\bar{\mathcal{P}})} |\ \iota(e) = l_{\bar{\theta}}^{(w,\bar{P})}\}$ and $\{f \in \mathcal{E}_{R(\theta,\mathcal{P})} |\ \iota(f) = l_\theta^{(w,P)}\}$, since the edges in each of these sets correspond in a one-to-one fashion with the distinct S-symbols $a \in \mathcal{F}(w)$. If e is the unique edge from $l_{\bar{\theta}}^{(w,\bar{P})}$ to $l_{\bar{\theta}}^{(wa,\bar{P})}$ labelled a, define $\nu_\beta(e) = f$, where f is the unique edge from $l_\theta^{(w,P)}$ to $l_\theta^{(wa,P)}$ labelled a. It is easy to see that ν_β defines a right resolving factor map $\Sigma_{R(\bar{\theta},\bar{\mathcal{P}})} \to \Sigma_{R(\theta,\mathcal{P})}$ and that $\pi_{r(\bar{\theta},\bar{\mathcal{P}})} = \pi_{r(\theta,\mathcal{P})}\nu_\beta$. ∎

We call ν_β the **factor map induced by** β. It can be shown, using Lemmas 6.1, 6.2 and 2.11, that the map ν_β does not depend on the choice of $P \in \mathcal{P}$. That is, if $Q \in \mathcal{P}$ and $\bar{Q} = \Lambda_\beta^{-1}(Q)$, then $l_{\bar\theta}^{(w,\bar{Q})} \to l_\theta^{(w,Q)}$ defines the same map ν_β. By Proposition 6.4, the map ν_β is unchanged if $\Sigma_{R(\bar\theta,\bar{\mathcal{P}})}$ and $\Sigma_{R(\theta,\mathcal{P})}$ are defined using an extension $\bar{m}$ of m.

PROPOSITION 3.8: *If β is a conjugacy, then ν_β is a conjugacy.*

Proof: Assume β is a conjugacy. Then there is a positive integer N such that if $x_1\cdots x_{2N}$ and $x_1'\cdots x_{2N}'$ are $\bar{A}$-words and $\beta(x_1\cdots x_{2N}) = \beta(x_1'\cdots x_{2N}')$, then $x_N = x_N'$. We claim that if $u_1\cdots u_{2N}$ and $u_1'\cdots u_{2N}'$ are $R(\bar\theta,\bar{\mathcal{P}})$-words such that $\nu_\beta(u_1\cdots u_{2N}) = \nu_\beta(u_1'\cdots u_{2N}')$ then $\tau(u_{2N}) = \tau(u_{2N}')$. This will show that ν_β is a conjugacy, since if $\nu_\beta(u) = \nu_\beta(u')$, then $\tau(u_i) = \tau(u_i')$ for all i, and therefore $u = u'$ since ν_β is right resolving. To show the claim, suppose that $\nu_\beta(u_1\cdots u_{2N}) = \nu_\beta(u_1'\cdots u_{2N}') = z_1\cdots z_{2N}$. Let $\iota(u_1) = l_{\bar\theta}^{(w,\bar{P})}$ and $\iota(u_1') = l_{\bar\theta}^{(v,\bar{P})}$. Then $l_\theta^{(w,P)} = \hat\nu_\beta(l_{\bar\theta}^{(w,\bar{P})}) = \hat\nu_\beta(l_{\bar\theta}^{(v,\bar{P})}) = l_\theta^{(v,P)}$. If $z_1\cdots z_{2N}$ is the unique path in $G(R(\theta,\mathcal{P}))$ from $l_\theta^{(w,P)} = l_\theta^{(v,P)}$ to $l_\theta^{(wa_1\cdots a_{2N},P)} = l_\theta^{(va_1\cdots a_{2N},P)}$ labelled $a_1\cdots a_{2N}$, then $u_1\cdots u_{2N}$ is the unique path in $G(R(\bar\theta,\bar{\mathcal{P}}))$ from $l_{\bar\theta}^{(w,\bar{P})}$ to $l_{\bar\theta}^{(wa_1\cdots a_{2N},\bar{P})}$ labelled $a_1\cdots a_{2N}$, and $u_1'\cdots u_{2N}'$ is the unique path from $l_{\bar\theta}^{(v,\bar{P})}$ to $l_{\bar\theta}^{(va_1\cdots a_{2N},\bar{P})}$ labelled $a_1\cdots a_{2N}$, by definition of ν_β. So $\tau(u_{2N}) = l_{\bar\theta}^{(wa_1\cdots a_{2N},\bar{P})}$ and $\tau(u_{2N}') = l_{\bar\theta}^{(va_1\cdots a_{2N},\bar{P})}$. To show these are the same, let $i \in S_{\bar{A}}$, and suppose that $(l_{\bar\theta}^{(wa_1\cdots a_{2N},\bar{P})})_i = 1$. Then there is an $\bar{A}$-word $\bar{p}x_1\cdots x_{2N} \in \bar\theta^{-1}(wa_1\cdots a_{2N})$, ending at i, with $\bar{p}_s \in \bar{P}$. Let $\beta(\bar{p}x_1\cdots x_{2N}) = py_1\cdots y_{2N}$. Then $p_s \in P$. Let $\tau(p) = j$. Since $p \in \theta^{-1}(w)$, we have $(l_\theta^{(w,P)})_j = 1$. Since $l_\theta^{(w,P)} = l_\theta^{(v,P)}$, we have $(l_\theta^{(v,P)})_j = 1$. So there is an A-word $q \in \theta^{-1}(v)$, ending at j, with $q_s \in P$. Therefore $qy_1\cdots y_{2N}$ is allowed. Let $\bar{q}x_1'\cdots x_{2N}' \in \beta^{-1}(qy_1\cdots y_{2N})$. Since $x_1\cdots x_{2N}$ and $x_1'\cdots x_{2N}'$ are in $\beta^{-1}(y_1\cdots y_{2N})$, it follows from the assumption on N that $x_N = x_N'$. Therefore $\bar{q}x_1'\cdots x_N' x_{N+1}\cdots x_{2N}$ is an allowed $\bar{A}$-word, ending at i, which is in $\bar\theta^{-1}(va_1\cdots a_{2N})$, with $\bar{q}_s \in \bar{P}$. Therefore $(l_{\bar\theta}^{(va_1\cdots a_{2N},\bar{P})})_i = 1$. A similar argument shows that if i is any state such that $(l_{\bar\theta}^{(va_1\cdots a_{2N},\bar{P})})_i = 1$, then $(l_{\bar\theta}^{(wa_1\cdots a_{2N},\bar{P})})_i = 1$. Therefore $\tau(u_{2N}) = l_{\bar\theta}^{(wa_1\cdots a_{2N},\bar{P})} = l_{\bar\theta}^{(va_1\cdots a_{2N},\bar{P})} = \tau(u_{2N}')$. This proves the claim, which completes the proof of the proposition. ∎

COROLLARY 3.9: *Let $\theta_1\colon \Sigma_{A_1} \to S$ and $\theta_2\colon \Sigma_{A_2} \to S$ be conjugate one-block factor maps, m a magic word for θ_1 and θ_2, and suppose that $\mathcal{P}_1$ is a θ_1-congruence partition and $\mathcal{P}_2$ is a θ_2-congruence partition, and that $\mathcal{P}_1$ is equivalent to*

$\mathcal{P}_2$. *Then there is a topological conjuacy* $\nu\colon \Sigma_{R(\theta_1,\mathcal{P}_1)} \to \Sigma_{R(\theta_2,\mathcal{P}_2)}$ *such that* $\pi_{r(\theta_1,\mathcal{P}_1)} = \pi_{r(\theta_2,\mathcal{P}_2)}\nu$.

Proof: Since $\mathcal{P}_1$ is equivalent to $\mathcal{P}_2$, there is a shift of finite type $\Sigma_{\bar{A}}$ and conjugacies $\beta_1\colon \Sigma_{\bar{A}} \to \Sigma_{A_1}$ and $\beta_2\colon \Sigma_{\bar{A}} \to \Sigma_{A_2}$ such that $\theta_1\beta_1 = \theta_2\beta_2 = \bar{\theta}$, and an extension $\bar{m}$ of m, such that $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1) = \Lambda_{\beta_2}^{-1}(\mathcal{P}_2) = \bar{\mathcal{P}}$. By Proposition 3.8, the maps $\nu_{\beta_1}\colon \Sigma_{R(\bar{\theta},\bar{\mathcal{P}})} \to \Sigma_{R(\theta_1,\mathcal{P}_1)}$ and $\nu_{\beta_2}\colon \Sigma_{R(\bar{\theta},\bar{\mathcal{P}})} \to \Sigma_{R(\theta_2,\mathcal{P}_2)}$ are conjugacies. Let $\nu = \nu_{\beta_2}\nu_{\beta_1}^{-1}$. By Lemma 3.7, $\pi_{r(\theta_2,\mathcal{P}_2)}\nu = \pi_{r(\theta_2,\mathcal{P}_2)}\nu_{\beta_2}\nu_{\beta_1}^{-1} = \pi_{r(\bar{\theta},\bar{\mathcal{P}})}\nu_{\beta_1}^{-1} = \pi_{r(\theta_1,\mathcal{P}_1)}$. ∎

We can now define the right closing cover induced by an equivalence class $[\mathcal{P}]$ of congruence partitions to be $\pi_{r(\theta,\mathcal{P})}\colon \Sigma_{R(\theta,\mathcal{P})} \to S$, where $\mathcal{P}$ is any representative. By Corollary 3.9, this is well-defined up to conjugacy of factor maps.

By the remarks following Lemma 2.13, there are only finitely many equivalence classes of congruence partitions for a given map θ. It follows from Corollary 3.9 that there are only finitely many conjugacy classes of right closing covers $\pi_{r(\theta,\mathcal{P})}\colon \Sigma_{R(\theta,\mathcal{P})} \to S$.

4. Decompositions of factor maps

In this section, we present our main results concerning decompositions of finite-to-one factor maps. The results are stated assuming that a given map θ can be decomposed as a left closing map followed by a right closing map. For each such result, there is a companion result, which assumes a decomposition of θ into a right closing map followed by a left closing map. The statements and proofs of these companion results should be clear to the reader.

In Lemmas 4.1, 4.3, and Definition 4.2, we will assume that Σ_A and Σ_B are irreducible shifts of finite type, S a sofic shift, that $\phi\colon \Sigma_A \to \Sigma_B$ is a left resolving factor map, $\gamma\colon \Sigma_B \to S$ is a right resolving factor map, and that $\theta = \gamma\phi$. Let m be a magic word for θ, with magic coordinate m_s, and let $\mathcal{P}$ be the partition induced by $\gamma\phi$. Let $P \in \mathcal{P}$. Since we will be dealing with a single map θ, we will write $l^{(w,P)} = l_\theta^{(w,P)}$.

LEMMA 4.1: *Let w be an S-word beginning with m, and suppose that $(l^{(w,P)})_i = 1$. If $j \in \mathcal{S}_A$, then $(l^{(w,P)})_j = 1$ if and only if $\hat{\phi}(i) = \hat{\phi}(j)$.*

Proof: Only if: Assume that w has length k. Suppose that $(l^{(w,P)})_j = 1$. Then there are A-words a, $b \in \theta^{-1}(w)$, ending at i and j respectively, such that

a_s, $b_s \in P$. Therefore $\phi(a_s) = \phi(b_s)$ by definition of $\mathcal{P}$. Since $\gamma(\phi(a)) = \gamma(\phi(b))$ and γ is right resolving, it follows that $\phi(a_s)\cdots\phi(a_k) = \phi(b_s)\cdots\phi(b_k)$. Since a ends at i and b ends at j, we have $\hat{\phi}(i) = \tau(\phi(a_k)) = \tau(\phi(b_k)) = \hat{\phi}(j)$.

If: Suppose that $\hat{\phi}(i) = \hat{\phi}(j)$. Since $(l^{(w,P)})_i = 1$, there is an A-word $a \in \theta^{-1}(w)$ ending at i with $a_s \in P$. So $\phi(a)$ is a B-word ending at $\hat{\phi}(i)$. Since $\hat{\phi}(i) = \hat{\phi}(j)$ and ϕ is left resolving, there is an A-word $c \in \phi^{-1}(\phi(a))$ ending at j. Since $\phi(a_s) = \phi(c_s)$, we have $a_s, c_s \in P$, by definition of $\mathcal{P}$. Since $\theta(c) = \gamma(\phi(c)) = \gamma(\phi(a)) = w$, we have $(l^{(w,P)})_j = 1$.

Definition 4.2: The vectors $l^{(w,P)}$ and $l^{(v,P)}$ are **disjoint** if the set of coordinates i such that $(l^{(w,P)})_i = 1$ is disjoint from the set of coordinates j such that $(l^{(v,P)})_j = 1$.

LEMMA 4.3: *Let w and v be S-words beginning with m. Then either $l^{(w,P)} = l^{(v,P)}$ or $l^{(w,P)}$ and $l^{(v,P)}$ are disjoint.*

Proof: Suppose that $l^{(w,P)}$ and $l^{(v,P)}$ are not disjoint. Then there is a coordinate i such that $(l^{(w,P)})_i = 1 = (l^{(v,P)})_i$. Then by Lemma 4.1, for any other $j \in \mathcal{S}_A$, we have $(l^{(w,P)})_j = 1$ if and only if $\hat{\phi}(i) = \hat{\phi}(j)$, if and only if $(l^{(v,P)})_j = 1$. Since $l^{(w,P)}$ and $l^{(v,P)}$ have non-zero components in exactly the same coordinates, we have $l^{(w,P)} = l^{(v,P)}$. ∎

LEMMA 4.4: *Let θ: $\Sigma_A \to S$ be a finite-to-one one-block factor map, $\mathcal{P}$ a congruence partition and $P \in \mathcal{P}$. Assume that for all S-words w and v beginning with m, either $l^{(w,P)} = l^{(v,P)}$, or $l^{(w,P)}$ and $l^{(v,P)}$ are disjoint. Then there is a left resolving factor map ρ: $\Sigma_A \to \Sigma_{R(\theta,\mathcal{P})}$ such that $\pi_{r(\theta,\mathcal{P})}\rho = \theta$.*

Proof: We first define ρ on states of Σ_A as follows. If $i \in \mathcal{S}_A$, choose an S-word w beginning with m such that $(l^{(w,P)})_i = 1$. Define $\hat{\rho}$: $\mathcal{S}_A \to \mathcal{S}_{R(\theta,\mathcal{P})}$ by $\hat{\rho}(i) = l^{(w,P)}$. To see that $\hat{\rho}$ is well-defined, note that if v is another word beginning with m and $(l^{(v,P)})_i = 1$, then $l^{(w,P)} = l^{(v,P)}$, since we are assuming that if $l^{(w,P)}$ and $l^{(v,P)}$ are not disjoint then they are equal. If e is an edge from i to j labelled a, we define $\rho(e) = f$, where f is the unique edge in $G(R(\theta,\mathcal{P}))$ from $l^{(w,P)}$ to $l^{(wa,P)}$ labelled a. To see that ρ is a factor map, observe that since $(l^{(w,P)})_i = 1$ and e is an edge from i to j labelled a, we have $(l^{(wa,P)})_j = 1$, and so $\hat{\rho}(j) = l^{(wa,P)}$. Therefore $\rho(e)$ is an edge from $\hat{\rho}(i)$ to $\hat{\rho}(j)$, and it follows that ρ is a factor map. Clearly $\pi_{r(\theta,\mathcal{P})}\rho = \theta$, since ρ preserves the labelling of edges.

We next show that ρ is left resolving. Suppose that $\rho(e_1) = \rho(e_2)$, where $\tau(e_1) = \tau(e_2) = j$. Assume that $\iota(e_1) = i_1$ and $\iota(e_2) = i_2$. Since $\hat{\rho}(i_1) = \hat{\rho}(i_2)$, there is an S-word w, beginning with m, such that $(l^{(w,P)})_{i_1} = 1 = (l^{(w,P)})_{i_2}$. Therefore, there are A-words $a, b \in \theta^{-1}(w)$, ending at i_1 and i_2, respectively, such that $a_s, b_s \in P$. Note that $\theta(e_1) = \theta(e_2)$, since $\theta = \pi_{r(\theta,\mathcal{P})}\rho$. Since Σ_A is transitive, there is an A-word g such that e_1ga is allowed (and therefore e_2ga is allowed, since $\tau(e_1) = \tau(e_2) = j$). Let $\theta(e_1g) = \theta(e_2g) = u$. Then ae_1ga and be_2ga are both in $\theta^{-1}(wuw)$, and so by Lemma 2.3, $a_s = b_s$. Now it follows from [KMT, Theorem 2.1] that $e_1 = e_2$, since otherwise θ would have a diamond. Therefore ρ is left resolving. ∎

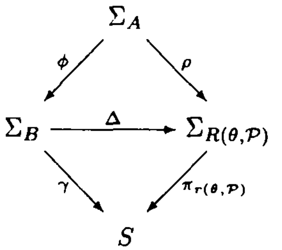
It follows from Lemmas 4.3 and 4.4 that if $\theta = \gamma\phi$, where ϕ is left resolving and γ is right resolving, then there is a map ρ as in Lemma 4.4.

Lemma 4.5: *Let Σ_A and Σ_B be irreducible shifts of finite type, S a sofic shift, and suppose that ϕ: $\Sigma_A \to \Sigma_B$ is a left resolving factor map, γ: $\Sigma_B \to S$ is a right resolving factor map, and that $\theta = \gamma\phi$. Let $\mathcal{P}$ be the partition induced by $\gamma\phi$, and let ρ be the map defined in Lemma 4.4. For $e_1, e_2 \in \mathcal{E}_A$, we have $\phi(e_1) = \phi(e_2)$ if and only if $\rho(e_1) = \rho(e_2)$.*

Proof: Suppose that $\phi(e_1) = \phi(e_2)$. Then $\theta(e_1) = \theta(e_2)$, since $\theta = \gamma\phi$. Let $\iota(e_1) = i_1$ and $\iota(e_2) = i_2$. Choose S-words w and v, beginning with m, such that $(l^{(w,P)})_{i_1} = 1$ and $(l^{(v,P)})_{i_2} = 1$. Since $\hat{\phi}(i_1) = \hat{\phi}(i_2)$, we have $(l^{(w,P)})_{i_2} = 1$ by Lemma 4.1. Therefore $l^{(w,P)}$ and $l^{(v,P)}$ are not disjoint, so they are equal, by Lemma 4.3. Therefore $\hat{\rho}(i_1) = \hat{\rho}(i_2)$ and so $\iota(\rho(e_1)) = \iota(\rho(e_2))$. Since $\pi_{r(\theta,\mathcal{P})}(\rho(e_1)) = \theta(e_1) = \theta(e_2) = \pi_{r(\theta,\mathcal{P})}(\rho(e_2))$, we have $\rho(e_1) = \rho(e_2)$, since $\pi_{r(\theta,\mathcal{P})}$ is right resolving.

Conversely, suppose that $\rho(e_1) = \rho(e_2)$. Then $\theta(e_1) = \theta(e_2)$, since $\theta = \pi_{r(\theta,\mathcal{P})}\rho$, by Lemma 4.4. If $\iota(e_1) = i_1$ and $\iota(e_2) = i_2$ and w and v are S-words, beginning with m, such that $(l^{(w,P)})_{i_1} = 1$ and $(l^{(v,P)})_{i_2} = 1$, then $l^{(w,P)} = \hat{\rho}(i_1) = \hat{\rho}(i_2) = l^{(v,P)}$. Therefore $(l^{(w,P)})_{i_2} = 1$ and so $\hat{\phi}(i_1) = \hat{\phi}(i_2)$, by Lemma 4.1. So $\iota(\phi(e_1)) = \iota(\phi(e_2))$. Since $\gamma(\phi(e_1)) = \theta(e_1) = \theta(e_2) = \gamma(\phi(e_2))$, we have $\phi(e_2) = \phi(e_1)$, since γ is right resolving. ∎

Lemma 4.6: *Let Σ_A and Σ_B be irreducible shifts of finite type, S a sofic shift, ϕ: $\Sigma_A \to \Sigma_B$ a left resolving factor map and γ: $\Sigma_B \to S$ a right resolving factor map, such that $\theta = \gamma\phi$. Let $\mathcal{P}$ be the partition induced by $\gamma\phi$. Then* *there is a graph isomorphism* $\Delta\colon G(B) \to G(R(\theta,\mathcal{P}))$, *which defines a conjugacy* $\Delta\colon \Sigma_B \to \Sigma_{R(\theta,\mathcal{P})}$, *such that the following diagram commutes.*



Proof: We define a graph isomorphism $\Delta\colon G(B) \to G(R(\theta,\mathcal{P}))$ as follows. Let $f \in \mathcal{E}_B$ and choose $e \in \phi^{-1}(f)$. Define $\Delta(f) = \rho(e)$. That Δ is well-defined follows from Lemma 4.5. If f_1f_2 is an allowed B-word, choose $e_1e_2 \in \phi^{-1}(f_1f_2)$. Then $\Delta(f_1f_2) = \rho(e_1e_2)$, which is an allowed $R(\theta,\mathcal{P})$- word, and so Δ defines a graph homomorphism. That Δ is an isomorphism follows immediately from Lemma 4.5. The commutativity of the diagram follows from the definition of Δ and the fact that $\gamma\phi = \pi_{r(\theta,\mathcal{P})}\rho$. ∎

THEOREM 4.7: *Let* $\theta\colon \Sigma_A \to S$ *be a finite-to-one factor map from an irreducible shift of finite type onto a sofic shift, and suppose that there is a shift of finite type* Σ_B, *a left closing factor map* $\phi\colon \Sigma_A \to \Sigma_B$ *and right closing factor map* $\gamma\colon \Sigma_B \to S$ *such that* $\gamma\phi = \theta$. *Let* $[\mathcal{P}]$ *be the partition induced by* $\gamma\phi$. *Then the factor map* $\gamma\colon \Sigma_B \to S$ *is conjugate to the right closing cover,* $\pi_{r(\theta,\mathcal{P})}\colon \Sigma_{R(\theta,\mathcal{P})} \to S$, *induced by* $[\mathcal{P}]$.

Proof: Recode ϕ to a left resolving factor map $\bar{\phi}\colon \Sigma_{\bar{A}} \to \Sigma_{\bar{B}}$, and γ to a right resolving factor map $\bar{\gamma}\colon \Sigma_{\bar{B}} \to S$ (see [K] or [BKM, Prop. 1]) Let $\bar{\theta} = \bar{\gamma}\bar{\phi}$ and let $\bar{\mathcal{P}}$ be the partition induced by $\bar{\gamma}\bar{\phi}$. By definition, $\bar{\mathcal{P}}$ is in the equivalence class $[\mathcal{P}]$. By Lemma 4.6, $\bar{\gamma}$ is conjugate to $\pi_{r(\bar{\theta},\bar{\mathcal{P}})}\colon \Sigma_{R(\bar{\theta},\bar{\mathcal{P}})} \to S$, which is, by definition, in the conjugacy class of the right closing cover induced by $[\mathcal{P}]$. ∎

If we assume that ϕ is right closing and γ is left closing, then we can conclude that γ is conjugate to the left closing cover $\pi_{l(\theta,\mathcal{P})}$. It is clear that for any decomposition $\gamma\phi$ as in Theorem 4.7, there exists a left closing factor map $\rho\colon \Sigma_A \to \Sigma_{R(\theta,\mathcal{P})}$ such that $\theta = \pi_{r(\theta,\mathcal{P})}\rho$ and $\gamma\phi$ is conjugate to $\pi_{r(\theta,\mathcal{P})}\rho$. If $\epsilon\colon \Sigma_B \to \Sigma_{R(\theta,\mathcal{P})}$ is a conjugacy such that $\pi_{r(\theta,\mathcal{P})}\epsilon = \gamma$, simply let $\rho = \epsilon\phi$.

THEOREM 4.8: *Let $\theta\colon \Sigma_A \to S$ be a finite-to-one factor map from an irreducible shift of finite type onto a sofic shift. Suppose that there are two decompositions of θ,*

$$\phi_1\colon \Sigma_A \to \Sigma_{B_1}, \gamma_1\colon \Sigma_{B_1} \to S \quad \text{and} \quad \phi_2\colon \Sigma_A \to \Sigma_{B_2}, \gamma_2\colon \Sigma_{B_2} \to S,$$

where Σ_{B_1} and Σ_{B_2} are shifts of finite type, ϕ_1 and ϕ_2 are left closing and γ_1 and γ_2 are right closing. Then $\gamma_1\phi_1$ and $\gamma_2\phi_2$ induce equivalent partitions if and only if $\gamma_1\phi_1$ and $\gamma_2\phi_2$ are conjugate decompositions.

Proof: The "if" direction follows from Proposition 2.14. To prove the converse, suppose that $\gamma_1\phi_1$ and $\gamma_2\phi_2$ induce equivalent partitions. By a theorem due to B. Kitchens (see [K] or [BKM, Prop. 1]), there exists a decomposition $\bar{\phi}_1\colon \Sigma_{A_1} \to \Sigma_{\bar{B}_1}, \bar{\gamma}_1\colon \Sigma_{\bar{B}_1} \to S$ which is conjugate to $\gamma_1\phi_1$, and a decomposition $\bar{\phi}_2\colon \Sigma_{A_2} \to \Sigma_{\bar{B}_2}, \bar{\gamma}_2\colon \Sigma_{\bar{B}_2} \to S$ which is conjugate to $\gamma_2\phi_2$, such that $\bar{\phi}_1$ and $\bar{\phi}_2$ are left resolving and $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are right resolving. Let $\theta_1 = \bar{\gamma}_1\bar{\phi}_1$ and $\theta_2 = \bar{\gamma}_2\bar{\phi}_2$. Let m be a magic word for θ_1 and θ_2, with magic coordinate m_s. Let $\mathcal{P}_1$ be the partition of $S(\theta_1, m)$ induced by $\bar{\gamma}_1\bar{\phi}_1$ and $\mathcal{P}_2$ be the partition of $S(\theta_2, m)$ induced by $\bar{\gamma}_2\bar{\phi}_2$. By assumption, $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent partitions.

Now, by Lemma 4.6, $\bar{\gamma}_1\bar{\phi}_1$ is conjugate to the decomposition $\rho_1\colon \Sigma_{A_1} \to \Sigma_{R(\theta_1,\mathcal{P}_1)}, \pi_{r(\theta_1,\mathcal{P}_1)}\colon \Sigma_{R(\theta_1,\mathcal{P}_1)} \to S$ and $\bar{\gamma}_2\bar{\phi}_2$ is conjugate to the decomposition $\rho_2\colon \Sigma_{A_2} \to \Sigma_{R(\theta_2,\mathcal{P}_2)}, \pi_{r(\theta_2,\mathcal{P}_2)}\colon \Sigma_{R(\theta_2,\mathcal{P}_2)} \to S$, where ρ_1 and ρ_2 are the factor maps defined in Lemma 4.4. It suffices to show that the $\pi_{r(\theta_1,\mathcal{P}_1)}\rho_1$ and $\pi_{r(\theta_2,\mathcal{P}_2)}\rho_2$ are conjugate decompositions. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are equivalent partitions, there exists a shift of finite type $\Sigma_{\bar{A}}$ and one-block conjugacies $\beta_1\colon \Sigma_{\bar{A}} \to \Sigma_{A_1}$ and $\beta_2\colon \Sigma_{\bar{A}} \to \Sigma_{A_2}$ such that $\theta_1\beta_1 = \theta_2\beta_2 = \bar{\theta}$, and an extension $\bar{m}$ of m which is magic for $\bar{\theta}$, such that $\Lambda_{\beta_1}^{-1}(\mathcal{P}_1) = \Lambda_{\beta_2}^{-1}(\mathcal{P}_2) = \bar{\mathcal{P}}$. To simplify notation, let us denote $\bar{m}$ by m.

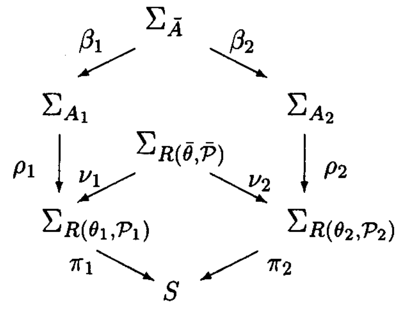
Let $\bar{P} \in \bar{\mathcal{P}}$ and let $P_1 = \Lambda_{\beta_1}(\bar{P})$ and $P_2 = \Lambda_{\beta_2}(\bar{P})$. Let

$$\nu_{\beta_1}\colon \Sigma_{R(\bar{\theta},\bar{\mathcal{P}})} \to \Sigma_{R(\theta_1,\mathcal{P}_1)} \quad \text{and} \quad \nu_{\beta_2}\colon \Sigma_{R(\bar{\theta},\bar{\mathcal{P}})} \to \Sigma_{R(\theta_2,\mathcal{P}_2)}$$

be the factor maps defined in Lemma 3.7. By Proposition 3.8, ν_{β_1} and ν_{β_2} are conjugacies. By Lemma 3.7, $\pi_{r(\theta_1,\mathcal{P}_1)}\nu_1 = \pi_{r(\bar{\theta},\bar{\mathcal{P}})} = \pi_{r(\theta_2,\mathcal{P}_2)}\nu_2$. To simplify notation, let us write

$$\nu_1 = \nu_{\beta_1}, \nu_2 = \nu_{\beta_2}, \pi_1 = \pi_{r(\theta_1,\mathcal{P}_1)}, \pi_2 = \pi_{r(\theta_2,\mathcal{P}_2)} \text{ and } \bar{\pi} = \pi_{r(\bar{\theta},\bar{\mathcal{P}})}.$$

CLAIM 1: *The following diagram commutes.*



Since $\beta_2\beta_1^{-1}$ and $\nu_2\nu_1^{-1}$ are conjugacies, the theorem will follow once we have established Claim 1. Since $\pi_1\nu_1 = \pi_2\nu_2$, by the previous remarks, it suffices to prove that $\nu_1^{-1}\rho_1\beta_1 = \nu_2^{-1}\rho_2\beta_2$. We will show that this equality follows from

CLAIM 2: *For any $\bar{A}$-word b there is an $R(\bar{\theta},\bar{\mathcal{P}})$- word f such that $\rho_1\beta_1(b) = \nu_1(f)$ and $\rho_2\beta_2(b) = \nu_2(f)$.*

To see that the equality $\nu_1^{-1}\rho_1\beta_1 = \nu_2^{-1}\rho_2\beta_2$ follows from Claim 2, let $x \in \Sigma_{\bar{A}}$ and let $b = x_{i-n}\dots x_{i+n}$ be a word appearing in x. If $f = f_{i-n}\dots f_{i+n}$ is the corresponding $R(\bar{\theta},\bar{\mathcal{P}})$-word from Claim 2, then since $\rho_1\beta_1(x_{i-n}\dots x_{i+n}) = \nu_1(f_{i-n}\dots f_{i+n})$ and ν_1 is a conjugacy, it follows that for sufficiently large n, we must have $[\nu_1^{-1}\rho_1\beta_1(x)]_i = f_i$. Similarly, $[\nu_2^{-1}\rho_2\beta_2(x)]_i = f_i$. Since i is arbitrary, we have $\nu_1^{-1}\rho_1\beta_1(x) = \nu_2^{-1}\rho_2\beta_2(x)$.

We now prove Claim 2, which will complete the proof of the theorem. Let b be a $\bar{A}$-word and let $\bar{\theta}(b) = z$. Choose $a \in \bar{\theta}^{-1}(m)$ such that $a_s \in \bar{P}$. Since $\Sigma_{\bar{A}}$ is irreducible, there exists an $\bar{A}$-word u such that aub is allowed. Let $\bar{\theta}(au) = w$, so that w is an S-word beginning with m. Clearly, $\bar{\theta}(aub) = wz$ is allowed. Let f be the unique path in $G(R(\bar{\theta},\bar{\mathcal{P}}))$ from $l_{\bar{\theta}}^{(w,\bar{P})}$ to $l_{\bar{\theta}}^{(wz,\bar{P})}$ labelled z. Similarly, let g be the unique path in $G(R(\theta_1,\mathcal{P}_1))$ from $l_{\theta_1}^{(w,P_1)}$ to $l_{\theta_1}^{(wz,P_1)}$ labelled z, and let h be the unique path in $G(R(\theta_2,\mathcal{P}_2))$ from $l_{\theta_2}^{(w,P_2)}$ to $l_{\theta_2}^{(wz,P_2)}$ labelled z. By definition of ν_1 and ν_2, we have $\nu_1(f) = g$ and $\nu_2(f) = h$. We will show that $\rho_1\beta_1(b) = g$ and $\rho_2\beta_2(b) = h$. First, we prove a

SUBCLAIM: $\iota(\rho_1\beta_1(b)) = l_{\theta_1}^{(w,P_1)} = \iota(g)$.

To see the subclaim, let $\iota(b) = j$ and $\hat{\beta}_1(j) = k$. Since au ends at j, it follows that $\beta_1(au) = c$ ends at k. Since $c \in \theta_1^{-1}(w)$ and $c_s \in P_1 = \Lambda_{\beta_1}(\bar{P})$, we have $(l_{\theta_1}^{(w,P_1)})_k = 1$. Therefore $\hat{\rho}_1(k) = l_{\theta_1}^{(w,P_1)}$, by definition of ρ_1 (see Lemma 4.4).

So we have $\iota(\rho_1\beta_1(b)) = \hat{\rho}_1\hat{\beta}_1(j) = \hat{\rho}_1(k) = l_{\theta_1}^{(w,P_1)} = \iota(g)$, which proves the subclaim.

Now, by the subclaim, $\rho_1\beta_1(b)$ and g are both in $(\pi_1)^{-1}(z)$ and both begin at $l_{\theta_1}^{(w,P_1)}$. Since π_1 is right resolving, by Lemma 3.7, it follows that $\rho_1\beta_1(b) = g$. A similar argument shows that $\rho_2\beta_2(b) = h$, which proves claim 2. This establishes the theorem. ∎

The theorem is also true if ϕ_1 and ϕ_2 are right closing and γ_1 and γ_2 are left closing. The proof is similar, and uses induced left closing covers.

COROLLARY 4.9: *Let θ: $\Sigma_A \to S$ be a finite-to-one factor map from an irreducible shift of finite type onto a sofic shift. Then there exist finitely many (possibly zero) conjugacy classes of decompositions of θ into a left closing map ϕ: $\Sigma_A \to \Sigma_B$ followed by a right closing map γ: $\Sigma_B \to S$ (where Σ_B is a shift of finite type). If $deg(\theta) = 1$, then any two decompositions of θ into a left closing map onto a shift of finite type followed by a right closing map are conjugate.*

Proof: By Theorem 4.8, if two such decompositions induce equivalent partitions, then they are conjugate. It follows that the number of conjugacy classes of decompositions is at most the number of equivalence classes of θ-congruence partitions, which is finite, by Lemma 2.13 and the remarks following it. The proof of the last statement follows by observing that if $deg(\theta) = 1$, then since $\#S(\theta, m) = 1$, there is only one partition of $S(\theta, m)$, and therefore just one equivalence class. ∎

The corollary is also true if we assume that ϕ is right closing and γ is left closing.

It is possible that a much stronger statement than Corollary 4.9 is true. In fact, it may be the case that there are only finitely many conjugacy classes of decompositions of θ into a finite number of factor maps $\Sigma_A \to \Sigma_{B_1} \to \dots \Sigma_{B_n} \to S$, where the intermediate shifts are of finite type, with no assumption that the maps are closing. An answer to this question, or even in the special case in which θ is closing, would be of some interest.

The following is an obvious consequence of Theorem 4.7 and the remarks following it.

COROLLARY 4.10: *Let θ: $\Sigma_A \to S$ be a finite-to-one factor map from an irreducible shift of finite type onto a sofic shift. Then there is a shift of finite type Σ_B,* *a left closing factor map* ϕ: $\Sigma_A \to \Sigma_B$ *and right closing factor map* γ: $\Sigma_B \to S$ *such that* $\gamma\phi = \theta$ *if and only if there is a* θ*-congruence partition* $[\mathcal{P}]$ *and a left closing factor map* ρ: $\Sigma_A \to \Sigma_{R(\theta,\mathcal{P})}$ *such that* $\pi_{r(\theta,\mathcal{P})}\rho = \theta$ *(where* $\pi_{r(\theta,\mathcal{P})}$ *is the right closing cover induced by* $[\mathcal{P}]$*).*

5. A decision procedure

In this section we show that if θ: $\Sigma_A \to S$ is a finite-to-one factor map, there is a finite procedure for finding all decompositions $\theta = \gamma\phi$, where γ is left closing and ϕ is right closing, up to conjugacy. By this, we mean that we can write down a finite list of decompositions such that any decomposition of θ into a left closing map (onto a shift of finite type) followed by a right closing map is conjugate to one in the list. In the process, we show that given any factor map θ: $\Sigma_A \to S$, not necessarily finite-to-one, and a finite-to-one factor map γ: $\Sigma_B \to S$, there is a finite procedure for finding all continuous, shift-commuting maps ϕ: $\Sigma_A \to \Sigma_B$ (not necessarily surjective) such that $\gamma\phi = \theta$.

We recall the definition of the fibered product [AKM, p. 487]. If θ: $\Sigma_A \to S$ and γ: $\Sigma_B \to S$ are factor maps, the **fibered product** of θ and γ is the shift of finite type $\Sigma_F = \{(x,y) \in \Sigma_A \times \Sigma_B: \theta(x) = \gamma(y)\}$. If θ and γ are one-block maps, then Σ_F is a one-step shift of finite type on the alphabet $\mathcal{E}_F = \{(a_1,a_2) \in \mathcal{E}_A \times \mathcal{E}_B: \theta(a_1) = \gamma(a_2)\}$, with transitions $(a_1,a_2) \to (a_1',a_2')$ if and only if $a_1 \to a_1'$ and $a_2 \to a_2'$. There are projections ψ_1: $\Sigma_F \to \Sigma_A$ and ψ_2: $\Sigma_F \to \Sigma_B$ defined by $\psi_1(x,y) = x$ and $\psi_2(x,y) = y$. It is not hard to see that if γ is finite-to-one, then so is ψ_1.

Proposition 5.1: *Let* θ: $\Sigma_A \to S$ *and* γ: $\Sigma_B \to S$ *be factor maps, where* γ *is finite-to-one. Let* ϕ: $\Sigma_A \to \Sigma_B$ *be continuous, shift-commuting map such that* $\gamma\phi = \theta$*. Then there exists an irreducible component* Σ_C *of* Σ_F*, the fibered product of* θ *and* γ*, such that the restriction of* ψ_1 *to* Σ_C *is a conjugacy and* $\phi = \psi_2(\psi_1|_{\Sigma_C})^{-1}$.

Proof: Let $\Sigma_C = \{(x,\phi(x)): x \in \Sigma_A\}$. Clearly, Σ_C is a subshift of Σ_F and $\psi_1|_{\Sigma_C}$: $\Sigma_C \to \Sigma_A$ is a conjugacy. We claim that Σ_C is an irreducible component of Σ_F. To see this, let $\Sigma_{C'}$ be the irreducible component of Σ_F containing Σ_C. Now ψ_1 is finite-to-one, since γ is, and so by [CP2, Corollary 4.4] we have $h(\Sigma_{C'}) = h(\Sigma_A)$. Clearly, $h(\Sigma_A) = h(\Sigma_C)$, and so $h(\Sigma_{C'}) = h(\Sigma_C)$. It now follows from [CP1, Theorem 3.3] that $\Sigma_C = \Sigma_{C'}$, which proves the claim. It is clear that $\phi = \psi_2(\psi_1|_{\Sigma_C})^{-1}$. ∎

Remarks 5.2: (i) It is well-known that there is a finite procedure for deciding whether a given factor map $f\colon \Sigma_C \to \Sigma_A$ between irreducible shifts of finite type is a conjugacy. To see this, assume that f has been recoded to a one-block map. By the proof of [KMT, Lemma 3.1], a one-block map f is a conjugacy if and only if it does not identify two distinct periodic points of period less than $(\#\mathcal{E}_C)^2 + N + 1$, where N is the least positive integer such that for any $i, j \in \mathcal{S}_C$, there is a path from i to j of length N (which exists since Σ_C is irreducible). Clearly this condition can be checked by a finite number of computations.

(ii) There is also a finite procedure for deciding whether a factor map $f\colon \Sigma_A \to \Sigma_B$ is right (or left) closing (see the proof of [K] or [BKM, Prop. 1]).

COROLLARY 5.3: *There is a finite procedure for determining all continuous, shift-commuting maps ϕ, as in Proposition 5.1. Furthermore, there is a finite procedure for determining all such right (or left) closing maps ϕ.*

Proof: We may assume that θ and γ are one-block maps. Then we can construct Σ_F as a one-step shift of finite type, which has finitely many components. For each such component Σ_C, there is a finite procedure for deciding whether $\psi_1|_{\Sigma_C}$ is a conjugacy, by Remark 5.2 (i). The last statement follows from Remark 5.2 (ii).

We can now combine this result with Theorem 4.7, to obtain the following result.

THEOREM 5.4: *Let $\theta\colon \Sigma_A \to S$ be a finite-to-one factor map from an irreducible shift of finite type onto a sofic shift. Then there is a finite procedure for determining all possible decompositions $\phi\colon \Sigma_A \to \Sigma_B$, $\gamma\colon \Sigma_B \to S$, where Σ_B is a shift of finite type, ϕ is left closing, γ is right closing, and $\gamma\phi = \theta$, up to conjugacy.*

Proof: We may assume that θ is a one-block map. Let C denote the collection of all right closing covers of the form $\pi_{r(\theta,\mathcal{P})}$, where $\mathcal{P}$ is a θ-congruence partition, and note that there is a finite procedure for constructing these. For each of these covers, there is a finite procedure for listing all left closing maps $\rho\colon \Sigma_A \to \Sigma_{R(\theta,\mathcal{P})}$ such that $\pi_{r(\theta,\mathcal{P})}\rho = \theta$, by Corollary 5.3. (Since θ is finite-to-one, it follows from [CP2, Corollary 4.4] and [CP1, Theorem 3.3] that any such map ρ must actually be surjective.) Let $\mathcal{D}$ denote the finite collection of all decompositions of θ which arise this way. By Theorem 4.7 and the remarks following, each decomposition of $\theta = \gamma\phi$ into a left closing map followed by a right closing map is conjugate to a decomposition $\rho: \Sigma_A \to \Sigma_{R(\theta,\mathcal{P})}$, $\pi_{r(\theta,\mathcal{P})}: \Sigma_{R(\theta,\mathcal{P})} \to S$, where ρ is left closing and $[\mathcal{P}]$ is the congruence partition induced by $\gamma\phi$. By Lemma 2.13 and Corollary 3.9, the cover $\pi_{r(\theta,\mathcal{P})}$ is conjugate to one of the covers in $\mathcal{C}$. It follows that the decomposition $\gamma\phi$ is conjugate to one of the decompositions in $\mathcal{D}$, and the result follows. ∎

It can similarly be shown that there is a finite procedure for finding all such decompositions where ϕ is right closing and γ is left closing.

6. More on the graphs $G(R(\theta,\mathcal{P}))$

In this section we show that for a given factor map $\theta: \Sigma_A \to S$, the graphs $G(R(\theta,\mathcal{P}))$ and $G(L(\theta,\mathcal{P}))$ are independent of the choice of $P \in \mathcal{P}$. As usual, we give the statements and proofs for $G(R(\theta,\mathcal{P}))$, those for $G(L(\theta,\mathcal{P}))$ being similar.

First, we alter our notation slightly. For given choices of magic word m, with magic coordinate m_s, a congruence partition $\mathcal{P}$ of $S(\theta,m)$ and $P \in \mathcal{P}$, we will denote the graph $G(R(\theta,\mathcal{P}))$ by $G(R(m,P))$. If $w = mv$ is a magic word beginning with m, we will write $l^{(w,P)} = l^{(mv,P)}$.

LEMMA 6.1: *If $\mathcal{P}$ is a congruence partition for m, and P, $P' \in \mathcal{P}$, then there is an S-word u such that mum is allowed and $\Gamma_{mum}(P) = P'$.*

Proof: Choose $\alpha \in P$ and $\alpha' \in P'$. Choose an A-word $a \in \theta^{-1}(m)$, with $a_s = \alpha$, and an A-word $c \in \theta^{-1}(m)$ with $c_s = \alpha'$. Since A is irreducible, there is an A-word b such that abc is allowed. Let $\theta(b) = u$. Then mum is allowed and $abc \in \theta^{-1}(mum)$, so that $\Gamma_{mum}(\alpha) = \alpha'$. Since $\mathcal{P}$ is a congruence partition, we must have $\Gamma_{mum}(P) = P'$. ∎

LEMMA 6.2: *Let u be a word such that mum is allowed. Let $\mathcal{P}$ be a congruence partition for m, $P \in \mathcal{P}$ and let $P' = \Gamma_{mum}(P)$. Then for any S-word $w \in \mathcal{F}(m)$, $l^{(mw,P')} = l^{(mumw,P)}$.*

Proof: Assume that m has length k. If $(l^{(mw,P')})_i = 1$, then there is an A-word $ab \in \theta^{-1}(mw)$, ending at i, such that $a_s \in P'$. Since Γ_{mum} is a bijection

$P \to P'$, there is an A-word $cd\bar{a} \in \theta^{-1}(mum)$ such that $c_t \in P$ and $\bar{a}_s = a_s$. Then the allowed A-word $cd\bar{a}_1 \ldots \bar{a}_{s-1}a_s \ldots a_k b \in \theta^{-1}(mumw)$ ends at i, so $(l^{(mumw,P)})_i = 1$. Conversely, if $(l^{(mumw,P)})_i = 1$, then there is an A word $cdab \in \theta^{-1}(mumw)$, ending at i, such that $c_t \in P$. Since $\Gamma_{mum}(P) = P'$, we have $a_s \in P'$, and so $(l^{(mw,P')})_i = 1$. ∎

PROPOSITION 6.3: *Let $\mathcal{P}$ be a congruence partition. If P, $P' \in \mathcal{P}$, then the labelled graphs $G(R(m,P))$ and $G(R(m,P'))$ are equal.*

Proof: By Lemma 6.1, there is an S-word u such that $\Gamma_{mum}(P) = P'$. Lemma 6.2 shows that any state of $G(R(m,P'))$ is a state of $G(R(m,P))$. To see the reverse inclusion, by Lemma 2.6 we can choose an S-word $\bar{u}$ such that $m\bar{u}m$ is allowed and $\Gamma_{m\bar{u}m} = \Gamma_{mum}^{-1}$. Then $\Gamma_{m\bar{u}m}(P') = P$, and for any word $w \in \mathcal{F}(m)$, $l^{(mw,P)} = l^{(mumw,P')}$, by Lemma 6.2. So the states of $G(R(m,P'))$ and $G(R(m,P))$ are equal, and the map which takes $l^{(mw,P')} \to l^{(mumw,P)}$ is the identity on states. Note that $\mathcal{F}(mw) = \mathcal{F}(mumw)$, since m is magic (see [W, Lemma 2.5]). Since there is a unique edge in $G(R(m,P'))$ from $l^{(mw,P')}$ to $l^{(mwa,P')}$ labelled a if and only if there is a unique edge edge in $G(R(m,P))$ from $l^{(mumw,P)}$ to $l^{(mumwa,P)}$ labelled a, the two labelled graphs are identical. ∎

It follows from Proposition 6.3 that the right closing cover $\pi_{r(\theta,\mathcal{P})}\colon \Sigma_{R(\theta,\mathcal{P})} \to S$ induced by $\mathcal{P}$ is independent of the choice of $P \in \mathcal{P}$. This is so because $\pi_{r(\theta,\mathcal{P})}$ is defined by the labelling of edges of $G(R(\theta,\mathcal{P}))$, which does not depend on P, by the previous lemma.

We next show that the graphs $G(R(\theta,\mathcal{P}))$ and $G(L(\theta,\mathcal{P}))$ are not changed if they are defined by an extension $\bar{m}$ of m.

PROPOSITION 6.4: *Let $\mathcal{P}$ be a congruence partition of $S(\theta,m)$, $P \in \mathcal{P}$, and let $\bar{m}$ be an extension of m. Then $\mathcal{P}$ is a congruence partition of $S(\theta,\bar{m})$ and the labelled graphs $G(R(\bar{m},P))$ and $G(R(m,P))$ are equal.*

Proof: Write $\bar{m} = umv$. It is easy to see from Definition 2.4 that for any S-word w such that $\bar{m}w\bar{m}$ is allowed, $\Gamma_{\bar{m}w\bar{m}} = \Gamma_{umvwumv} = \Gamma_{mvwum}$, since the beginning u and the ending v have no effect on the definition of $\Gamma_{umvwumv}$. Therefore $\Gamma_{\bar{m}w\bar{m}}(\mathcal{P}) = \Gamma_{mwm}(\mathcal{P}) = \mathcal{P}$, since $\mathcal{P}$ is a congruence partition for m, so $\mathcal{P}$ is a congruence partition for $\bar{m}$ and for mv.

To see that $G(R(\bar{m},P)) = G(R(m,P))$, observe that for any word $w \in \mathcal{F}(\bar{m})$, we have $l^{(\bar{m}w,P)} = l^{(umvw,P)} = l^{(mvw,P)}$, since the beginning u has no effect on the definition of $l^{(umvw,P)}$. It follows that $G(R(\bar{m},P)) = G(R(mv,P))$. Now, choose an S-word n such that $mvnm$ is allowed. Let $\Gamma_{mvnm}(P) = P'$. For any $w \in \mathcal{F}(m)$, we have $l^{(mvnmw,P)} = l^{(mw,P')}$ by Lemma 6.2. By an argument similar to the proof of Proposition 6.3, it follows that $G(R(mv,P)) = G(R(m,P'))$. Since $G(R(m,P')) = G(R(m,P))$, by Proposition 6.3, we have $G(R(\bar{m},P)) = G(R(m,P))$. ∎

It can be shown that a different choice of magic word m' generates the same collection of graphs $G(R(m',P))$ and $G(L(m',P))$.

Finally, we show that the shifts of finite type $\Sigma_{R(\theta,\mathcal{P})}$ are irreducible.

Proposition 6.5: *If $\mathcal{P}$ is a θ-congruence partition, then the shifts of finite type $\Sigma_{R(\theta,\mathcal{P})}$ and $\Sigma_{L(\theta,\mathcal{P})}$ are irreducible.*

Proof: Let $P \in \mathcal{P}$ and let $l^{(mw,P)}$ and $l^{(mv,P)}$ be states of $G(R(\theta,\mathcal{P}))$. Since S is irreducible, there is a word u such that $mvum$ is allowed. By Lemma 2.6, there is a word u' such that $mvumu'm$ is allowed and $\Gamma_{mvumu'm}$ is the identity on $S(m,\theta)$. Then $\Gamma_{mvumu'm}(P) = P$. Write $umu'mw = a_1\cdots a_r$. By Lemma 6.2, $l^{(mw,P)} = l^{(mva_1\cdots a_r,P)}$. Therefore $l^{(mv,P)}, l^{(mva_1,P)}, \ldots, l^{(mva_1\cdots a_r,P)}$ is a path in $G(R(\theta,\mathcal{P}))$ from $l^{(mv,P)}$ to $l^{(mw,P)}$. Since $l^{(mw,P)}$ and $l^{(mv,P)}$ are arbitrary, we have shown that $\sum_{R(\theta,\mathcal{P})}$ is irreducible. A similar argument shows that $\Sigma_{L(\theta,\mathcal{P})}$ is irreducible. ∎

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